

On Algebraic Semigroups and Monoids, II

Michel Brion

Abstract

Consider an algebraic semigroup S and its closed subscheme of idempotents, $E(S)$. When S is commutative, we show that $E(S)$ is finite and reduced; if in addition S is irreducible, then $E(S)$ is contained in a smallest closed irreducible subsemigroup of S , and this subsemigroup is an affine toric variety. It follows that $E(S)$ (viewed as a partially ordered set) is the set of faces of a rational polyhedral convex cone. On the other hand, when S is an irreducible algebraic monoid, we show that $E(S)$ is smooth, and its connected components are conjugacy classes of the unit group.

1 Introduction

This article continues the study of algebraic semigroups (not necessarily linear), began in [Br12, BrRe12]. The idempotents play an essential rôle in the structure of abstract semigroups; by results of [loc. cit.], the idempotents of algebraic semigroups satisfy remarkable existence and finiteness properties. In this article, we consider the subscheme of idempotents, $E(S)$, of an algebraic semigroup S over an algebraically closed field; we show that $E(S)$ has a very special structure under additional assumptions on S . Our first main result states:

Theorem 1.1. *Let M be an irreducible algebraic monoid, and G its unit group. Then the scheme $E(M)$ is smooth, and its connected components are conjugacy classes of G .*

Note that the scheme of idempotents of an algebraic semigroup is not necessarily smooth. Consider indeed an arbitrary variety X equipped with the composition law $(x, y) \mapsto x$; then X is an algebraic semigroup, and $E(X)$ is the whole X . Yet the scheme of idempotents is reduced for all examples that we know of; it is tempting to conjecture that $E(S)$ is reduced for any algebraic semigroup S .

When S is commutative, $E(S)$ turns out to be a combinatorial object, as shown by our second main result:

Theorem 1.2. *Let S be a commutative algebraic semigroup. Then the scheme $E(S)$ is finite and reduced. If S is irreducible, then $E(S)$ is contained in a smallest closed irreducible subsemigroup of S ; moreover, this subsemigroup is a toric monoid.*

By a toric monoid, we mean an irreducible algebraic monoid M with unit group being a torus; then M is affine, as follows e.g. from [Ri07, Thm. 2]. Thus, M may be viewed as an

affine toric variety (not necessarily normal). Conversely, every such variety has a unique structure of algebraic monoid that extends the multiplication of its open torus (see e.g. [Ri98, Prop. 1]). So we may identify the toric monoids with the affine toric varieties. Toric monoids have been studied by Putcha under the name of connected diagonal monoids (see [Pu81]); they have also been investigated by Neeb in [Ne92].

In view of Theorem 1.2 and of the structure of toric monoids, the set of idempotents of any irreducible commutative algebraic semigroup, equipped with its natural partial order, is isomorphic to the poset of faces of a rational polyhedral convex cone.

Theorem 1.2 extends readily to the case where S has a dense subsemigroup generated by a single element; then S is commutative, but not necessarily irreducible. Thereby, one associates an affine toric variety with any point of an algebraic semigroup; the corresponding combinatorial data may be seen as weak analogues of the spectrum of a linear operator (see Example 3.9 for details). This construction might deserve further study.

This article is organized as follows. In Subsection 2.1, we present simple proofs of some basic results, first obtained in [Br12, BrRe12] by more complicated arguments; also, we prove the first assertion of Theorem 1.2. Subsection 2.2 investigates the local structure of an algebraic semigroup at an idempotent, in analogy with the Peirce decomposition,

$$R = eRe \oplus (1 - e)Re \oplus eR(1 - e) \oplus (1 - e)R(1 - e),$$

of a ring R equipped with an idempotent e . As an application, we show that the isolated idempotents of an irreducible algebraic semigroup are exactly the central idempotents (Proposition 2.10). In Subsection 2.3, we obtain a slightly stronger version of Theorem 1.1, by combining our local structure analysis with results of Putcha on irreducible linear algebraic monoids (see [Pu88, Chap. 6]). As an application, we generalize Theorem 1.1 to the intervals in $E(S)$, where S is an irreducible algebraic semigroup in characteristic zero (Corollary 2.17).

We return to commutative semigroups in Subsection 3.1, and show that every irreducible commutative algebraic semigroup has a largest closed toric submonoid (Proposition 3.2). The structure of toric monoids is recalled in Subsection 3.2, and Theorem 1.2 is proved in the case of such monoids. The general case is deduced in Subsection 3.3, which also contains applications to algebraic semigroups having a dense cyclic subsemigroup (Corollary 3.7).

In the final Subsection 3.4, we consider those irreducible algebraic semigroups S such that $E(S)$ is finite. We first show how to reduce their structure to the case where S is a monoid and has a zero; then S is linear in view of [BrRi07, Cor. 3.3]). Then we present another proof of a result of Putcha: any irreducible algebraic monoid having a zero and finitely many idempotents must have a solvable unit group (see [Pu82, Cor. 10], and [Pu88, Prop. 6.24] for a generalization). Putcha also showed that any irreducible linear algebraic monoid with nilpotent unit group has finitely many idempotents, but this does not extend to solvable unit groups (see [Pu81, Thm. 1.12, Ex. 1.15]). We refer to work of Huang (see [Hu96a, Hu96b]) for further results on irreducible linear algebraic monoids having finitely many idempotents.

Notation and conventions. Throughout this article, we consider varieties and schemes over a fixed algebraically closed field k . We use the textbook [Ha77] as a general reference for algebraic geometry. Unless otherwise stated, schemes are assumed to be separated and of finite type over k ; a *variety* is a reduced scheme (in particular, varieties are not necessarily irreducible). By a *point* of a variety X , we mean a k -rational point; we identify X with its set of points equipped with the Zariski topology and with the structure sheaf.

An *algebraic semigroup* is a variety S equipped with an associative composition law $\mu : S \times S \rightarrow S$. For simplicity, we denote $\mu(x, y)$ by xy for any $x, y \in S$. A point $e \in S$ is *idempotent* if $e^2 = e$. The set of idempotents is equipped with a partial order \leq defined by $e \leq f$ if $e = ef = fe$. Also, the idempotents are the k -rational points of a closed subscheme of S : the scheme-theoretic preimage of the diagonal under the morphism $S \rightarrow S \times S$, $x \mapsto (x^2, x)$. We denote that subscheme by $E(S)$.

An *algebraic monoid* is an algebraic semigroup M having a neutral element, 1_M . The *unit group* of M is the subgroup of invertible elements, $G(M)$; this is an algebraic group, open in M (see [Ri98, Thm. 1] in the case where M is irreducible; the general case follows easily, see [Br12, Thm. 2.2.4]).

We shall address some rationality questions for algebraic semigroups, and use [Sp98, Chap. 11] as a general reference for basic rationality results on varieties. As in [loc. cit.], we fix a subfield F of k , and denote by F_s the separable closure of F in k ; the Galois group of F_s over F is denoted by Γ . We say that an algebraic semigroup S is *defined over* F , if the variety S and the morphism $\mu : S \times S \rightarrow S$ are both defined over F .

2 The idempotents of an algebraic semigroup

2.1 Existence

We first obtain a simple proof of the following basic result ([Br12, Prop. 2.1.6], proved there by reducing to a finite field):

Proposition 2.1. *Let S be an algebraic semigroup. Then S has an idempotent.*

Proof. Arguing by noetherian induction, we may assume that S has no proper closed subsemigroup. As a consequence, the set of powers x^n , where $n \geq 1$, is dense in S for any $x \in S$; in particular, S is commutative. Also, yS is dense in S for any $y \in S$; since yS is constructible, it contains a nonempty open subset of S . Thus, there exists $n = n(x, y) \geq 1$ such that $x^n \in yS$.

Choose $x \in S$. For any $n \geq 1$, let

$$S_n := \{y \in S \mid x^n \in yS\}.$$

Then each S_n is a constructible subset of S , since S_n is the image of the closed subset $\{(y, z) \in S \times S \mid yz = x^n\}$ under the first projection. Moreover, $S = \bigcup_{n \geq 1} S_n$.

To show that the closed subscheme $E(S)$ is nonempty, we may replace k with any larger algebraically closed field, and hence assume that k is uncountable. Then, by the next lemma, there exists $n \geq 1$ such that S_n contains a nonempty open subset of S . Since the set of powers x^{mn} , where $m \geq 1$, is dense in S , it follows that S_n contains some x^{mn} .

Equivalently, there exists $z \in S$ such that $x^n = x^{mn}z$. Let $y := x^n$, then $y = y^mz$ and hence $y^{m-1} = y^{2m-2}z$. Thus, $y^{m-1}z$ is idempotent. \square

Lemma 2.2. *Let X be a variety, and $(X_i)_{i \in I}$ a countable family of constructible subsets such that $X = \bigcup_{i \in I} X_i$. If k is uncountable, then some X_i contains a nonempty open subset of X .*

Proof. Since each X_i is constructible, it can be written as a finite disjoint union of irreducible locally closed subsets. We may thus assume that each X_i is locally closed and irreducible; then we may replace X_i with its closure, and thus assume that all the X_i are closed and irreducible. We may also replace X with any nonempty open subset U , and X_i with $X_i \cap U$. Thus, we may assume in addition that X is irreducible. We then have to show that $X_i = X$ for some $i \in I$.

We now argue by induction on the dimension of X . If $\dim(X) = 1$, then each X_i is either a finite subset or the whole X . But the X_i cannot all be finite: otherwise, X , and hence k , would be countable. This yields the desired statement.

In the general case, assume that each X_i is a proper subset of X . Since the set of irreducible hypersurfaces in X is uncountable, there exists such a hypersurface Y which is not contained in any X_i . In other words, each $Y \cap X_i$ is a proper subset of Y . Since $Y = \bigcup_{i \in I} Y \cap X_i$, applying the induction assumption to Y yields a contradiction. \square

Next, we obtain refinements of [Br12, Prop. 2.3.2(iii), Prop. 3.5.1(ii)], thereby proving the first assertion of Theorem 1.2:

Proposition 2.3. *Let S be a commutative algebraic semigroup.*

- (i) *The scheme $E(S)$ is finite and reduced.*
- (ii) *S has a smallest idempotent, e_0 .*
- (iii) *If the algebraic semigroup S is defined over F , then so is e_0 .*

Proof. (i) It suffices to show that the Zariski tangent space $T_e(E(S))$ is zero for any idempotent e . Since $E(S) = \{x \in S \mid x^2 = x\}$ and S is commutative, we obtain

$$T_e(E(S)) = \{z \in T_e(S) \mid 2f(z) = z\},$$

where f denotes the tangent map at e of the multiplication by e in S (see Lemma 2.5 below for details on the determination of $T_e(E(S))$ when S is not necessarily commutative). Moreover, f is an idempotent endomorphism of the vector space $T_e(S)$, and hence is diagonalizable with eigenvalues 0 and 1. This yields the desired vanishing of $T_e(E(S))$.

(ii) By (i), the subscheme $E(S)$ consists of finitely many points e_1, \dots, e_n of S . Their product, $e_1 \cdots e_n =: e_0$, satisfies $e_0^2 = e_0$ and $e_0 e_i = e_0$ for $i = 1, \dots, n$. Thus, e_0 is the smallest idempotent.

(iii) Assume that the variety S and the morphism μ are defined over F ; then $E(S)$ is a F -subscheme of S , and hence a smooth F -subvariety by (i). In view of [Sp98, Thm. 11.2.7], it follows that $E(S)$ (regarded as a finite subset of $S(k)$) is contained in $S(F_s)$; also, $E(S)$ is clearly stable by Γ . Thus, $e_0 \in S(F_s)$ is invariant under Γ , and hence $e_0 \in S(F)$. \square

We now deduce from Proposition 2.3 another fundamental existence result (which also follows from [BrRe12, Thm. 2.1]):

Corollary 2.4. *Let S be an algebraic semigroup defined over F . If S has an F -rational point, then it has an F -rational idempotent.*

Proof. Let $x \in S(F)$ and denote by $\langle x \rangle$ the smallest closed subsemigroup of S containing x . Then $\langle x \rangle$ is the closure of the set of powers x^n , where $n \geq 1$. Thus, $\langle x \rangle$ is a commutative algebraic semigroup, defined over F . So $\langle x \rangle$ contains an idempotent defined over F , by the previous proposition. \square

2.2 Local structure

In this subsection, we fix an algebraic semigroup S and an idempotent $e \in S$. Then e defines two endomorphisms of the variety S : the left multiplication, $e_\ell : x \mapsto ex$, and the right multiplication, $e_r : x \mapsto xe$. Clearly, these endomorphisms are commuting idempotents, i.e., they satisfy $e_\ell^2 = e_\ell$, $e_r^2 = e_r$, and $e_\ell e_r = e_r e_\ell$. Since e_ℓ and e_r fix the point e , their tangent maps at that point are commuting idempotent endomorphisms, f_ℓ and f_r , of the Zariski tangent space $T_e(S)$. Thus, we have a decomposition into joint eigenspaces

$$T_e(S) = T_e(S)_{0,0} \oplus T_e(S)_{1,0} \oplus T_e(S)_{0,1} \oplus T_e(S)_{1,1}, \quad (1)$$

where we set

$$T_e(S)_{a,b} := \{z \in T_e(S) \mid f_\ell(z) = az, f_r(z) = bz\}$$

for $a, b = 0, 1$. The Zariski tangent space of $E(S)$ at e has a simple description in terms of these eigenspaces:

Lemma 2.5. *With the above notation, we have*

$$T_e(E(S)) = T_e(S)_{1,0} \oplus T_e(S)_{0,1}. \quad (2)$$

Moreover, $T_e(E(S))$ is the image of $f_r - f_\ell$.

Proof. We claim that

$$T_e(E(S)) = \{z \in T_e(S) \mid f_\ell(z) + f_r(z) = z\}.$$

Indeed, recall that $E(S)$ is the preimage of the diagonal under the morphism $S \rightarrow S \times S$, $x \mapsto (\text{sq}(x), x)$, where $\text{sq} : S \rightarrow S$, $x \mapsto x^2$ denotes the square map. Thus, we have

$$T_e(E(S)) = \{z \in T_e(S) \mid T_e(\text{sq})(z) = z\},$$

where $T_e(\text{sq})$ denotes the tangent map of sq at e . Also, sq is the composition of the diagonal morphism, $\delta : S \rightarrow S \times S$, followed by the multiplication, $\mu : S \times S \rightarrow S$. Thus, we have

$$T_e(\text{sq}) = T_{(e,e)}(\mu) \circ T_e(\delta)$$

with an obvious notation. Furthermore, $T_e(\delta) : T_e(S) \rightarrow T_e(S) \times T_e(S)$ is the diagonal embedding; also, $T_{(e,e)}(\mu) : T_e(S) \times T_e(S) \rightarrow T_e(S)$ equals $f_\ell \times f_r$, since the restriction of μ to $\{e\} \times S$ (resp. $S \times \{e\}$) is just e_ℓ (resp. e_r). Thus, $T_e(\text{sq}) = f_\ell + f_r$; this proves the claim.

Now (2) follows readily from the claim together with the decomposition (1). For the second assertion, let $z \in T_e(S)$ and write $z = z_{0,0} + z_{1,0} + z_{0,1} + z_{1,1}$ in that decomposition. Then $(f_r - f_\ell)(z) = z_{0,1} - z_{1,0}$ and hence $\text{Im}(f_r - f_\ell) = T_e(S)_{1,0} \oplus T_e(S)_{0,1}$. \square

Next, we observe that each joint eigenspace of f_ℓ and f_r in $T_e(S)$ is the Zariski tangent space to a naturally defined closed subsemigroup scheme of S . Consider indeed the closed subscheme

$${}_eS_e := \{x \in S \mid ex = xe = e\},$$

where the right-hand side is understood as the scheme-theoretic fiber at e of the morphism $e_\ell \times e_r : S \rightarrow S \times S$. Define similarly

$$eS_e := \{x \in S \mid ex = x, xe = e\}, \quad {}_eSe := \{x \in S \mid ex = e, xe = x\},$$

and finally

$$eSe := \{x \in S \mid ex = xe = x\}.$$

Then one readily obtains:

Lemma 2.6. *With the above notation, ${}_eS_e$, eS_e , ${}_eSe$, and eSe are closed subsemigroup schemes of S containing e . Moreover, we have*

$$T_e({}_eS_e) = T_e(S)_{0,0}, \quad T_e(eS_e) = T_e(S)_{1,0}, \quad T_e({}_eSe) = T_e(S)_{0,1}, \quad T_e(eSe) = T_e(S)_{1,1}.$$

Remarks 2.7. (i) Note that ${}_eS_e$ is the largest closed subsemigroup scheme of S containing e as its zero. This subscheme is not necessarily reduced, as shown e.g. by [Br12, Ex. 3.2.4]. Specifically, consider the affine space \mathbb{A}^3 equipped with pointwise multiplication; this is a toric monoid. Let M be the hypersurface of \mathbb{A}^3 with equation

$$z^n - xy^n = 0,$$

where n is a positive integer. Then M is a closed toric submonoid, containing $e := (1, 0, 0)$ as an idempotent. Moreover, ${}_eM_e$ is the closed subscheme of \mathbb{A}^3 with ideal generated by $x - 1$ and $z^n - y^n$. Thus, ${}_eM_e$ is everywhere nonreduced whenever n is a multiple of the characteristic of k (assumed to be nonzero).

(ii) Also, note that eS_e is the largest closed subsemigroup scheme of S containing e and such that the composition law is the second projection. (Indeed, every such subsemigroup scheme S' satisfies $ex = x$ and $xe = e$ for any T -valued point x of S' , where T is an arbitrary scheme; in other words, $S' \subset eS_e$. Conversely, for any T -valued points x, y of eS_e , we have $xy = xey = ey = y$). In particular, eS_e consists of idempotents, and $eS_e = xS_x$ for any k -rational point x of eS_e .

Likewise, ${}_eSe$ is the largest closed subsemigroup scheme of S containing e and such that the composition law is the first projection. We shall see in Corollary 2.9 that ${}_eSe$ and eS_e are in fact reduced.

(iii) Finally, eSe is the largest closed submonoid scheme of S with neutral element e . This subscheme is reduced, since it is the image of the morphism $S \rightarrow S$, $x \mapsto exe$. Likewise, Se and eS are closed subsemigroups of S , and

$$T_e(Se) = T_e(S)_{0,1} \oplus T_e(S)_{1,1}, \quad T_e(eS) = T_e(S)_{1,0} \oplus T_e(S)_{1,1}.$$

One would like to have a ‘global’ analogue of the decomposition (1) along the lines of the local structure results for algebraic monoids obtained in [Br08] (which makes an essential use of the unit group). Specifically, one would like to describe some open neighborhood of e in S by means of the product of the four pieces ${}_eS_e$, ${}_eSe$, eS_e , and eSe (taken in a suitable order) and of the composition law of S . But this already fails when S is commutative: then both ${}_eSe$ and eS_e consist of the reduced point e , so that we only have two nontrivial pieces, S_e and eS ; moreover, the restriction of the composition law to $S_e \times eS \rightarrow S$ is just the second projection, since $xy = xey = ey = y$ for all $x \in S_e$ and $y \in eS$. Yet we shall obtain global analogues of certain partial sums in the decomposition (1). For this, we introduce some notation.

Consider the algebraic monoid eSe and its unit group, $G(eSe)$. Since $G(eSe)$ is open in eSe , the set

$$U = U(e) := \{x \in S \mid exe \in G(eSe)\} \quad (3)$$

is open in S . Clearly, U contains e and is stable under e_ℓ and e_r ; also, note that

$$Ue \cap eU = eUe = G(eSe).$$

We now describe the structure of Ue :

Lemma 2.8. *Keep the above notation.*

- (i) $Ue = \{x \in Se \mid ex \in G(eSe)\}$ and ${}_eUe = {}_eSe$.
- (ii) Ue is an open subsemigroup of Se .
- (iii) The morphism

$$\varphi : {}_eSe \times G(eSe) \longrightarrow S, \quad (x, g) \longmapsto xg$$

is a locally closed immersion with image Ue . Moreover, φ is an isomorphism of semigroup schemes, where the composition law of the left-hand side is given by $(x, g)(y, h) := (x, gh)$.

- (iv) The tangent map of φ at (e, e) induces an isomorphism

$$T_e(S)_{0,1} \oplus T_e(S)_{1,1} \cong T_e(Ue) = T_e(Se).$$

Proof. Both assertions of (i) are readily checked. The first assertion implies that Ue is open in Se . To show that Ue is a subsemigroup, note that for any points x, y of Ue , we have $exy = exey$. Hence $exy \in G(eSe)$ by (i), so that $xy \in Ue$ by (i) again. This completes the proof of (ii).

For (iii), consider T -valued points x of ${}_eSe$ and g of $G(eSe)$, where T is an arbitrary scheme. Then $exge = ege = g$ and hence xg is a T -valued point of Ue . Thus, φ yields a morphism ${}_eSe \times G(eSe) \rightarrow Ue$. Moreover, we have

$$\varphi(x, g)\varphi(y, h) = xgyh = xgeyh = xgeh = xgh = \varphi((x, g)(y, h)),$$

that is, φ is a homomorphism of semigroup schemes. To show that φ is an isomorphism, consider a T -valued point z of Ue and denote by $(ez)^{-1}$ the inverse of ez in $G(eSe)$. Then $z = xg$, where $x := z(ez)^{-1}$ and $g := ez$; moreover, $x \in ({}_eSe)(T)$ and $g \in (eSe)(T)$. Also, if $z = yh$ where $y \in ({}_eSe)(T)$ and $h \in (eSe)(T)$, then $h = eh = eyh = ez$ and $y = ye = yhh^{-1} = z(ez)^{-1}$. Thus, the morphism

$$Ue \longrightarrow {}_eSe \times G(eSe), \quad z \longmapsto (z(ez)^{-1}, ez)$$

is the inverse of φ .

Finally, (iv) follows readily from (iii) in view of Lemma 2.6. \square

Corollary 2.9. (i) *The scheme ${}_eSe$ is reduced, and is a union of connected components of $E(Se)$.*

(ii) *If S is irreducible, then ${}_eSe$ is the unique irreducible component of $E(Se)$ through e .*

Proof. Since Ue is reduced, and isomorphic to ${}_eSe \times G(eSe)$ in view of Lemma 2.8, we see that ${}_eSe$ is reduced as well. Moreover, that lemma also implies that ${}_eSe = E(Se) \cap Ue$ (as schemes). In particular, ${}_eSe$ is open in $E(Se)$. But ${}_eSe$ is also closed; this proves (i).

Next, assume that S is irreducible; then so are Se and Ue . By Lemma 2.8 again, ${}_eSe$ is irreducible as well, which implies (ii). \square

Note that a dual version of Lemma 2.8 yields the structure of eU ; also, eS_e satisfies the dual statement of Corollary 2.9.

We now obtain a description of the isolated points of $E(S)$ (viewed as a topological space). To state our result, denote by $C = C(e)$ the union of those irreducible components of S that contain e , or alternatively, the closure of any neighborhood of e in S ; then C is a closed subsemigroup of S .

Proposition 2.10. *With the above notation, e is isolated in $E(S)$ if and only if e centralizes C ; then $E(S)$ is reduced at e .*

In particular, the isolated idempotents of an irreducible algebraic semigroup are exactly the central idempotents.

Proof. Assume that e centralizes C ; then e_ℓ and e_r induce the same endomorphism of the local ring $\mathcal{O}_{C,e} = \mathcal{O}_{S,e}$. Thus, $f_\ell = f_r$. By Lemma 2.5, it follows that $T_e(E(S)) = \{0\}$. Hence e is an isolated reduced point of $E(S)$.

Conversely, if e is isolated in $E(S)$, then it is also isolated in ${}_eSe$ and in eS_e (since they both consist of idempotents). As ${}_eSe$ and eS_e are reduced, it follows that $T_e({}_eSe) = \{0\} = T_e(eS_e)$, i.e., $T_e(S)_{0,1} = T_e(S)_{1,0} = \{0\}$. In view of Lemma 2.5, we thus have $T_e(E(S)) = \{0\}$, i.e., $E(S)$ is reduced at e . Moreover, $f_\ell = f_r$ by Lemma 2.5 again. In other words, e_ℓ and e_r induce the same endomorphism of $\mathfrak{m}/\mathfrak{m}^2$, where \mathfrak{m} denotes the maximal ideal of $\mathcal{O}_{S,e}$. Hence e_ℓ and e_r induce the same endomorphism of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ for any integer $n \geq 1$, since the natural map $\text{Sym}^n(\mathfrak{m}/\mathfrak{m}^2) \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective and equivariant for the natural actions of e_ℓ and e_r . Next, consider the endomorphisms of $\mathcal{O}_{S,e}/\mathfrak{m}^n$ induced by e_r and e_ℓ : these are commuting idempotents of this finite-dimensional k -vector space, which preserve the filtration by the quotients $\mathfrak{m}^m/\mathfrak{m}^n$ ($0 \leq m \leq n$) and coincide on the associated graded vector space. Thus, $e_\ell = e_r$ as endomorphisms of $\mathcal{O}_{S,e}/\mathfrak{m}^n$ for all n , and hence as endomorphisms of $\mathcal{O}_{S,e}$. This means that for any $f \in \mathcal{O}_{S,e}$ there exists a neighborhood $V = V_f$ of e in S such that $f(ex) = f(xe)$ for all $x \in V$. Since $\mathcal{O}_{S,e}$ is the localization of a finitely generated k -algebra, it follows that we may choose V independently of f . Then $xe = ex$ for all $x \in V$, and hence for all $x \in C$, since V is dense in C . \square

We now return to the decomposition (1), and obtain a global analogue of the partial sum $T_e(S)_{1,0} \oplus T_e(S)_{0,1} \oplus T_e(S)_{1,1}$ in terms of the open subset U :

Lemma 2.11. (i) *The morphism*

$$\psi : {}_eSe \times G(eSe) \times eS_e \longrightarrow S, \quad (x, g, y) \longmapsto xgy$$

is a locally closed immersion with image UeU .

(ii) *The tangent map of ψ at (e, e, e) induces an isomorphism*

$$T_e(S)_{0,1} \oplus T_e(S)_{1,1} \oplus T_e(S)_{1,0} \cong T_e(UeU).$$

Proof. (i) Let $z \in UeU$; then $z = z'z''$ with $z' \in Ue$ and $z'' \in eU$. In view of Lemma 2.8, it follows that $z = xgy$ with $x \in {}_eSe$, $g \in G(eSe)$, and $y \in eS_e$. Then $ze = xge = xg$; likewise, $ez = egy = gy$. Thus, we have $g = eze$, $x = ze(eze)^{-1}$, and $y = (eze)^{-1}ez$. In particular, z satisfies the following conditions: $z \in U$, and $z = ze(eze)^{-1}(eze)(eze)^{-1}ez$. Conversely, if $z \in S$ satisfies the above two conditions, then $z \in {}_eSeG(eSe)eS_e \subset (Ue)(eU) = UeU$. Also, these conditions clearly define a locally closed subset of S . This yields the assertions.

(ii) follows from (i) in view of Lemma 2.6. \square

Finally, we obtain a parameterization of those idempotents of S that are contained in UeU . To state it, let

$$V = V(e) := \{(x, y) \in {}_eSe \times eS_e \mid yx \in G(eSe)\}.$$

Then V is an open neighborhood of (e, e) in ${}_eSe \times eS_e$. For any point (x, y) of V , we denote by $(yx)^{-1}$ the inverse of yx in $G(eSe)$.

Lemma 2.12. *With the above notation, the morphism*

$$\gamma : V \longrightarrow S, \quad (x, y) \longmapsto x(yx)^{-1}y$$

induces an isomorphism from V to the scheme-theoretic intersection $UeU \cap E(S)$. Moreover, the tangent map of γ at (e, e) induces an isomorphism

$$T_{0,1}(S) \oplus T_{1,0}(S) \cong T_e(UeU \cap E(S)).$$

Proof. We argue with T -valued points for an arbitrary scheme T , as in the proof of Lemma 2.8 (iii).

Let $z \in UeU$. By Lemma 2.11, we may write z uniquely as xgy , where $x \in {}_eSe$, $g \in G(eSe)$, and $y \in eS_e$. If $z \in E(S)$, then of course $xgyxgy = xgy$. Multiplying by e on the left and right, this yields $gyxg = g$ and hence $gyx = e$. Thus, $(x, y) \in V$ and $z = \gamma(x, y)$. Conversely, if $(x, y) \in V$, then $x(yx)^{-1} \in {}_eSeG(eSe)$ and hence $x(yx)^{-1} \in Ue$ by Lemma 2.8. Using that lemma again, it follows that $\gamma(x, y) \in UeU$. Also, one readily checks that $\gamma(x, y)$ is idempotent. This shows the first assertion, which in turn implies the second assertion. \square

Remarks 2.13. (i) The above subsets Ue , eU , UeU , and eUe are contained in the corresponding equivalence classes of e under Green's relations (see e.g. [Pu88, Def. 1.1] for the definition of these relations).

Indeed, for any $x \in Ue$, we have obviously $S^1x \subset S^1e = Se$, where S^1 denotes the monoid obtained from S by adjoining a neutral element. Also, S^1x contains $(ex)^{-1}ex = e$. Thus, $S^1x = S^1e$, that is, $x\mathcal{L}e$ with the notation of [loc. cit.]. Likewise, $x\mathcal{R}e$ for any $x \in eU$, and $x\mathcal{J}e$ for any $x \in UeU$. Finally, $eUe = G(eSe)$ equals the \mathcal{H} -equivalence class of e .

Also, one readily checks that eS_e (resp. ${}_eS_e$) is the set of idempotents in the equivalence class of e under \mathcal{R} (resp. \mathcal{L}).

(ii) Consider the centralizer of e in S ,

$$C_S(e) := \{x \in S \mid xe = ex\}.$$

This is a closed subsemigroup scheme of S containing both eSe and ${}_eS_e$. Moreover, the (left or right) multiplication by e yields a retraction of semigroup schemes $C_S(e) \rightarrow eSe$, and we have

$$T_e C_S(e) = T_e(S)_{0,0} \oplus T_e(S)_{1,1}.$$

We may also consider the left centralizer of e in S ,

$$C_S^\ell(e) := \{x \in S \mid ex = exe\}.$$

This is again a closed subsemigroup scheme of S , which contains both Se and ${}_eS_e$. Also, one readily checks that the left multiplication e_ℓ yields a retraction of semigroup schemes $C_S^\ell(e) \rightarrow eSe$, and we have

$$T_e C_S^\ell(e) = T_e(S)_{0,0} \oplus T_e(S)_{0,1} \oplus T_e(S)_{1,1}.$$

Moreover, $C_S^\ell(e) \cap U$ is the preimage of $G(eSe)$ under e_ℓ .

The right centralizer of e in S ,

$$C_S^r(e) := \{x \in S \mid xe = exe\},$$

satisfies similar properties; note that $C_S(e) = C_S^\ell(C_S^r(e)) = C_S^r(C_S^\ell(e))$. Also, one easily checks that U is stable under $C_S^\ell(e) \times C_S^r(e)$ acting on S by left and right multiplication.

(iii) Recall the description of Green's relations for an algebraic monoid M with dense unit group G (see [Pu84, Thm. 1]). For any $x, y \in M$, we have:

$$\begin{aligned} x\mathcal{L}y &\Leftrightarrow \overline{Mx} = \overline{My} \Leftrightarrow Gx = Gy, & x\mathcal{R}y &\Leftrightarrow \overline{xM} = \overline{yM} \Leftrightarrow xG = yG, \\ x\mathcal{J}y &\Leftrightarrow \overline{MxM} = \overline{MyM} \Leftrightarrow GxG = GyG. \end{aligned}$$

(Indeed, $x\mathcal{L}y$ if and only if $Mx = My$; then $\overline{Mx} = \overline{My}$. Since Gx is the unique dense open G -orbit in \overline{Mx} , it follows that $Gx = Gy$. Conversely, if $Gx = Gy$ then $Mx = My$. This proves the first equivalence; the next ones are checked similarly).

In view of (i), it follows that $Ue \subset Ge$ for any idempotent e of M . Likewise, $eU \subset eG$ and hence $UeU \subset GeG$; also, $eUe \subset eGe$. These inclusions are generally strict, e.g., when M is the monoid of $n \times n$ matrices and $e \neq 0, 1$. Thus, Ue is in general strictly contained in the \mathcal{L} -class of e , and likewise for eU , UeU .

Also, in view of (ii), U is stable under left multiplication by $C_G^\ell(e)$ and right multiplication by $C_G^r(e)$. In particular, Ue is an open subset of Me containing $C_G^\ell(e)e$. If M is irreducible, then $C_G^\ell(e)e$ is open in Me by [Pu88, Thm. 6.16 (ii)]. As a consequence, U contains the open $C_G^\ell(e) \times C_G^r(e)$ -stable neighborhood M_0 of e in M , whose structure is described in [Br08, Thm. 2.2.1]. Yet M_0 is in general strictly contained in U .

2.3 Smoothness

In this subsection, we first obtain a slight generalization of Theorem 1.1; we then apply this result to intervals in idempotents of irreducible algebraic semigroups.

Recall that an algebraic monoid M is *unit dense* if it is the closure of its unit group; this holds e.g. when M is irreducible. We may now state:

Theorem 2.14. *Let M be a unit dense algebraic monoid, G its unit group, and T a maximal torus of G ; denote by M° (resp. G°) the neutral component of M (resp. of G), and by \overline{T} the closure of T in M . Then the scheme $E(M)$ is smooth, and equals $E(M^\circ)$. Moreover, the connected components of $E(M)$ are conjugacy classes of G° ; every such component meets \overline{T} .*

Proof. Consider an idempotent $e \in M$, and its open neighborhood U defined by (3). Then UeU is a locally closed subvariety of M by Lemma 2.11. We claim that UeU is smooth at e .

To prove the claim, note that the subset GeG of M is a smooth, locally closed subvariety, since it is an orbit of the algebraic group $G \times G$ acting on M by left and right multiplication. Also, $GeG \supset UeU \supset (G \cap U)e(G \cap U)$, where the first inclusion follows from Remark 2.13 (iii). Moreover, $G \cap U$ is an open neighborhood of e , dense in U (as G is dense in M). Since the orbit map $G \times G \rightarrow GeG$, $(x, y) \mapsto xey$ is flat, it follows that $(G \cap U)e(G \cap U)$ is an open neighborhood of e in GeG , and hence in UeU . This yields the claim.

By that claim together with Lemma 2.11, the schemes ${}_eMe$ and eM_e are smooth at e . In view of Lemma 2.12, it follows that e is contained in a smooth, locally closed subvariety V of $E(M)$ such that

$$\dim_e(V) = \dim_e({}_eMe) + \dim_e(eM_e).$$

Using Lemmas 2.5 and 2.6, we obtain

$$\dim_e(V) = \dim T_e(M)_{0,1} + \dim T_e(M)_{1,0} = \dim T_e(E(M)).$$

Thus, V contains an open neighborhood of e in $E(M)$; in particular, $E(M)$ is smooth at e . We have shown that the scheme $E(M)$ is smooth.

Next, recall that M° is a closed irreducible submonoid of M with unit group G° (see [Br12, Prop. 2.2.1, Prop. 2.4.3]). Also, $E(M) = E(M^\circ)$ as sets, in view of [loc. cit., Rem. 3.2.8 (ii)]. Since $E(M)$ is smooth, it follows that $E(M) = E(M^\circ)$ as schemes.

To complete the proof, we may replace M with M° and hence assume that M is irreducible; then G is connected. Thus, G has a largest closed connected affine normal subgroup, G_{aff} (see e.g. [Ro56, Thm. 16, p. 439]). Denote by M_{aff} the closure of G_{aff} in M . By [Br12, Thm. 3.3.4], M_{aff} is an irreducible affine algebraic monoid with unit group G_{aff} , and $E(M) = E(M_{\text{aff}})$ as sets. Thus, we may further assume that M is affine, or equivalently linear (see [Pu88, Thm. 3.15]). Then every conjugacy class in $E(M)$ meets \overline{T} by [loc. cit., Cor. 6.10]. Moreover, $E(\overline{T})$ is finite by [loc. cit., Thm. 8.4] (or alternatively by Proposition 2.3). Thus, it suffices to check that the G -conjugacy class of every $e \in E(\overline{T})$ is closed in M .

This assertion is shown in [Br08, Lem. 1.2.3] under the additional assumption that k has characteristic 0. Yet that assumption is unnecessary; we recall the argument for the

convenience of the reader. Let B be a Borel subgroup of G containing T ; since G/B is complete, it suffices to show that the B -conjugacy class of e is closed in M . But that class is also the U -conjugacy class of e , where U denotes the unipotent part of B ; indeed, we have $B = UT$, and T centralizes e . So the desired closedness assertion follows from the fact that all orbits of a unipotent algebraic group acting on an affine variety are closed (see e.g. [Sp98, Prop. 2.4.14]). \square

Remarks 2.15. (i) If S is a *smooth* algebraic semigroup, then the scheme $E(S)$ is smooth as well. Indeed, for any idempotent e of S , the variety Se is smooth (since it is the image of the smooth variety S under the retraction e_r). In view of Lemma 2.8, it follows that ${}_eSe$, and likewise eS_e , are smooth at e . This implies in turn that $E(S)$ is smooth at e , by arguing as in the third paragraph of the proof of Theorem 2.14.

(ii) In particular, the scheme of idempotents of any finite-dimensional associative algebra A is smooth. This can be proved directly as follows. Firstly, one reduces to the case of an unital algebra: consider indeed the algebra $B := k \times A$, where the multiplication is given by $(t, x)(u, y) := (tu, ty + ux + xy)$. Then B is a finite-dimensional associative algebra with unit $(1, 0)$. Moreover, one checks that the scheme $E(B)$ is the disjoint union of two copies of $E(A)$: the images of the morphisms $x \mapsto (0, x)$ and $x \mapsto (1, -x)$. Secondly, if A is unital with unit group G , and $e \in A$ is idempotent, then the tangent map at 1 of the orbit map

$$G \longrightarrow A, \quad g \longmapsto geg^{-1}$$

is identified with $f_r - f_\ell$ under the natural identifications of $T_1(G)$ and $T_e(A)$ with A . In view of Lemma 2.5, it follows that the conjugacy class of e contains a neighborhood of e in $E(A)$; this yields the desired smoothness assertion.

(iii) One may ask for a simpler proof of Theorem 2.14 based on a tangent map argument as above. But in the setting of that theorem, there seems to be no relation between the Zariski tangent spaces of M at the smooth point 1_M and at the (generally singular) point e .

Still considering a unit dense algebraic monoid M with unit group G , we now describe the isotropy group scheme of any idempotent $e \in M$ for the G -action by conjugation, i.e., the centralizer $C_G(e)$ of e in G . Recall from Remark 2.13 (ii) that the centralizer of e in M is equipped with a retraction of monoid schemes $\tau : C_M(e) \rightarrow eMe$; thus, τ restricts to a retraction of group schemes that we still denote by $\tau : C_G(e) \rightarrow G(eMe)$. We may now state the following result, which generalizes [Br08, Lem. 1.2.2 (iii)] with a more direct proof:

Proposition 2.16. *With the above notation, we have an exact sequence of group schemes*

$$1 \longrightarrow {}_eG_e \longrightarrow C_G(e) \xrightarrow{\tau} G(eMe) \longrightarrow 1.$$

Proof. Clearly, the scheme-theoretic kernel of $\tau : C_G(e) \rightarrow G(eMe)$ equals ${}_eG_e$. Since $G(eMe)$ is reduced, it remains to show that τ is surjective on k -rational points. For this, consider the left stabilizer $C_M^\ell(e)$ equipped with its reduced subscheme structure. This is a closed submonoid of M ; moreover, the map

$$\tau^\ell : C_M^\ell(e) \longrightarrow eMe, \quad x \longmapsto ex$$

is a retraction and a homomorphism of algebraic monoids (see Remark 2.13 (ii) again). Also, $C_G^\ell(e) := C_M^\ell(e) \cap G$ is a closed subsemigroup of G , and hence a closed subgroup by [Re05, Exc. 3.5.1.2]. This yields a homomorphism of algebraic groups that we still denote by $\tau^\ell : C_G^\ell(e) \rightarrow G(eMe)$.

We claim that the latter homomorphism is surjective. Indeed, let $x \in G(eMe)$. Then $xM = eM$, since $x \in eM$ and $e = xx^{-1} \in xM$. As xG is the unique dense G -orbit in xM for the G -action on M by right multiplication, it follows that $xG = eG$. Hence there exists $g \in G$ such that $x = eg$; then $ege = xe = x = eg$, i.e., $x \in C_G^\ell(e)$. This proves the claim.

Next, observe that $\overline{C_G^\ell(e)}$ (closure in M) is a unit dense submonoid of M . Moreover, with the notation of Theorem 2.14, we have $e \in \overline{T}$ and hence $T \subset C_G(e) \subset C_G^\ell(e)$; thus, $e \in \overline{C_G^\ell(e)}$. Therefore, $G(eMe) = eC_G^\ell(e)$ is contained in $\overline{C_G^\ell(e)}$ as well. Thus, to show the desired surjectivity, we may replace M with $\overline{C_G^\ell(e)}$. Then we apply the claim to the right stabilizer $C_G^r(e)$; this yields the statement, since $C_G^r(C_G^\ell(e)) = C_G(e)$. \square

Finally, we apply Theorem 2.14 to the structure of intervals in $E(S)$, where S is an algebraic semigroup. Given two idempotents $e_0, e_1 \in S$ such that $e_0 \leq e_1$, we consider

$$[e_0, e_1] := \{x \in E(S) \mid e_0 \leq x \leq e_1\}.$$

This has a natural structure of closed subscheme of S , namely, the scheme-theoretic intersection $E(S) \cap {}_{e_0}S_{e_0} \cap e_1Se_1$ (since $e_0 \leq x$ if and only if $x \in {}_{e_0}S_{e_0}$, and $x \leq e_1$ if and only if $x \in e_1Se_1$). Note that e_1Se_1 is a closed submonoid of S containing e_0 . Moreover,

$${}_{e_0}S_{e_0} \cap e_1Se_1 = {}_{e_0}(e_1Se_1)_{e_0} =: M(e_0, e_1) = M \quad (4)$$

is a closed submonoid scheme of e_1Se_1 with zero e_0 , and $[e_0, e_1] = E(M)$ as schemes. We may now state the following result, which sharpens and builds on a result of Putcha (see [Pu88, Thm. 6.7]):

Corollary 2.17. *Keep the above notation, and assume that k has characteristic 0 and S is irreducible. Then M is reduced, affine, and unit dense. Moreover, the interval $[e_0, e_1]$ is smooth; each connected component of $[e_0, e_1]$ is a conjugacy class of $G(M)^\circ$.*

Proof. We may replace S with e_1Se_1 ; thus, we may assume that S is an irreducible algebraic monoid, and $e_1 = 1_S$. Then $M = {}_{e_0}S_{e_0}$ is reduced, as follows from the local structure of S at e_0 (see [Br08]); more specifically, an open neighborhood of e_0 in M is isomorphic to a homogeneous fiber bundle with fiber ${}_{e_0}M_{e_0}$ by [loc. cit., Thm. 2.2.1, Rem. 3.1.3]. Moreover, M is affine and unit dense by [loc. cit., Lem. 3.1.4]. The final assertion follows from these results in view of Theorem 2.14. \square

Note that the above statement does not extend to positive characteristics. Indeed, M can be nonreduced in that case, as shown by the example in Remark 2.7 (i).

3 Irreducible commutative algebraic semigroups

3.1 The finite poset of idempotents

Throughout this subsection, we consider an irreducible commutative algebraic semigroup S . We first record the following easy result:

Proposition 3.1. *S has a largest idempotent, e_1 . Moreover, there exists a positive integer n such that $x^n \in e_1 S$ for all $x \in S$. If S is defined over F , then so is e_1 .*

Proof. By the finiteness of $E(S)$ (Proposition 2.3) together with [BrRe12, Cor. 2.6], there exist a positive integer n , an idempotent $e_1 \in S$, and a nonempty open subset U of S such that x^n is a unit of the closed submonoid $e_1 S$ for all $x \in U$. Since S is irreducible, it follows that $x^n \in e_1 S$ for all $x \in S$. In particular, $e \in e_1 S$ for any $e \in E(S)$, i.e., $e = e_1 e$; hence e_1 is the largest idempotent.

If S is defined over F , then e_1 is defined over F_s (by Proposition 2.3 again) and is clearly invariant under Γ . Thus, $e_1 \in E(S)(F)$. \square

With the above notation, the unit group $G(e_1 S)$ is a connected commutative algebraic group, and hence has a largest subtorus, T . The closure \overline{T} of T in S is a closed toric submonoid with neutral element e_1 and unit group T . We now gather further properties of \overline{T} :

Proposition 3.2. *With the above notation, we have:*

- (i) $E(S) = E(\overline{T})$.
- (ii) \overline{T} contains every subtorus of S .
- (iii) If S is defined over F , then so is \overline{T} .

Proof. (i) Note that $\overline{T} = e_1 \overline{T}$, and $E(S) = e_1 E(S) = E(e_1 S)$. Thus, we may replace S with $e_1 S$, and hence assume that S is an irreducible commutative algebraic monoid. Then the first assertion follows from Theorem 2.14.

(ii) Let S' be a subtorus of S (i.e., a locally closed subsemigroup which is isomorphic to a torus as an algebraic semigroup), and denote by e the neutral element of S' . Then $S' = e S' \subset e S = e_1 e S \subset e_1 S$. Hence we may again replace S with $e_1 S$, and assume that S is an irreducible commutative algebraic monoid. Now consider the map

$$\varphi : S \longrightarrow eS, \quad x \longmapsto xe.$$

Then φ is a surjective homomorphism of algebraic monoids. Thus, φ restricts to a homomorphism of unit groups $G := G(S) \rightarrow G(eS)$; the image of that homomorphism is closed, and also dense since G is dense in S . Thus, φ sends G onto $G(eS)$. Since $S' = G(eS')$ is a closed connected subgroup of $G(eS)$, there exists a closed connected subgroup G' of G such that $\varphi(G') = S'$. Let G'_{aff} denote the largest closed connected affine subgroup of G' . Since S' is affine, we also have $\varphi(G'_{\text{aff}}) = S'$, as follows from the decomposition $G' = G'_{\text{aff}} G'_{\text{ant}}$, where G'_{ant} denotes the largest closed subgroup of G such that every homomorphism from G'_{ant} to an affine algebraic group is constant (see e.g. [Ro56, Cor. 5, pp. 440–441]). But in view of the structure of commutative affine algebraic groups (see e.g. [Sp98, Thm. 3.1.1]), we have $G'_{\text{aff}} = T' \times U'$, where T' (resp. U') denotes the largest subtorus (resp. the largest

unipotent subgroup) of G'_{aff} . Thus, $\varphi(T') = S'$, that is, $S' = eT'$. Since $T' \subset T$ and $e \in \overline{T}$, it follows that $S' \subset \overline{T}$.

(iii) Since e_1 is defined over F , so is e_1S . Hence the algebraic group $G(e_1S)$ is also defined over F , by [Sp98, Prop. 11.2.8 (ii)]. In view of [SGA3, Exp. XIV, Thm. 1.1], it follows that the largest subtorus T of $G(e_1S)$ is defined over F as well. Thus, so is \overline{T} . \square

3.2 Toric monoids

As mentioned in the introduction, the toric monoids are exactly the affine toric varieties (not necessarily normal). For later use, we briefly discuss their structure; details can be found e.g. in [Ne92], [Pu81, §2, §3], and [Re05, §3.3].

The isomorphism classes of toric monoids are in a bijective correspondence with the pairs (Λ, \mathcal{M}) , where Λ is a free abelian group and \mathcal{M} is a finitely generated submonoid of Λ which generates that group. This correspondence assigns to a toric monoid M with unit group T , the lattice Λ of characters of T and the monoid \mathcal{M} of weights of T acting on the coordinate ring $\mathcal{O}(M)$ via its action on M by multiplication. Conversely, one assigns to a pair (Λ, \mathcal{M}) , the torus $T := \text{Hom}(\Lambda, \mathbb{G}_m)$ (consisting of all group homomorphisms from Λ to the multiplicative group) and the monoid $M := \text{Hom}(\mathcal{M}, \mathbb{A}^1)$ (consisting of all monoid homomorphisms from \mathcal{M} to the affine line equipped with the multiplication). The coordinate ring $\mathcal{O}(T)$ (resp. $\mathcal{O}(M)$) is identified with the group ring $k[\Lambda]$ (resp. the monoid ring $k[\mathcal{M}]$).

Via this correspondence, the idempotents of M are identified with the monoid homomorphisms $\varepsilon : \mathcal{M} \rightarrow \{1, 0\}$. Any such homomorphism is uniquely determined by the preimage of 1, which is the intersection of \mathcal{M} with a unique face of the cone, $C(\mathcal{M})$, generated by \mathcal{M} in the vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Moreover, every T -orbit in M (for the action by multiplication) contains a unique idempotent. This defines bijective correspondences between the idempotents of M , the faces of the rational, polyhedral, convex cone $C(\mathcal{M})$, and the T -orbits in M . Via these correspondences, the partial order relation on $E(M)$ is identified with the inclusion of faces, resp. of orbit closures; moreover, the dimension of a face is the dimension of the corresponding orbit. In particular, there is a unique closed orbit, corresponding to the minimal idempotent and to the smallest face of $C(\mathcal{M})$ (which is also the largest linear subspace contained in $C(\mathcal{M})$).

The toric monoid M is defined over F if and only if Λ is equipped with a continuous action of Γ that stabilizes \mathcal{M} (this is shown e.g. in [Sp98, Prop. 3.2.6] for tori; the case of toric monoids is handled by similar arguments). Under that assumption, the above correspondences are compatible with the actions of Γ .

We also record the following observation:

Lemma 3.3. *Let M be a toric monoid, and S a closed irreducible subsemigroup of M . Then S is a toric monoid.*

Proof. Since S is irreducible, there is a unique T -orbit in M that contains a dense subset of S . Thus, there exists a unique idempotent e_S of M such that $S \cap e_S T$ is dense in S . Then $S \cap e_S T$ is a closed irreducible subsemigroup of the torus $e_S T$, and hence a subtorus in view of [Re05, Exc. 3.5.1.2]. This yields our assertion. \square

Next, consider a toric monoid M with unit group T and smallest idempotent e_0 , so that the closed T -orbit in M is e_0T . Denote by T_0 the reduced neutral component of the isotropy group T_{e_0} , and by $\overline{T_0}$ the closure of T_0 in M . Clearly, $\overline{T_0}$ is a toric monoid with torus T_0 and zero e_0 ; this monoid is the reduced scheme of the monoid scheme $M(e_0, e_1)$ defined by (4). We now gather further properties of $\overline{T_0}$:

Proposition 3.4. *With the above notation, we have:*

- (i) $\dim(T_0)$ equals the length of any maximal chain of idempotents in M .
- (ii) $\overline{T_0}$ is the smallest closed irreducible subsemigroup of M containing $E(M)$.
- (iii) $\overline{T_0}$ is the largest closed irreducible subsemigroup of M having a zero.
- (iv) If M is defined over F , then so is $\overline{T_0}$.

Proof. (i) Since e_0T is closed in M , we easily obtain that $e_0M = e_0T$. Also, e_0T is isomorphic to the homogeneous space T/T_{e_0} , where T_{e_0} denotes the (scheme-theoretic) stabilizer of e_0 in T . Thus, the morphism $\varphi : M \rightarrow e_0M$, $x \mapsto e_0x$ makes M a T -homogeneous fiber bundle over T/T_{e_0} ; its scheme-theoretic fiber at e_0 is the closure of T_{e_0} in M . Therefore, we have

$$\dim(M) - \dim(e_0M) = \dim(T_{e_0}) = \dim(T_0).$$

Thus, $\dim(T_0)$ is the codimension of the smallest face of the cone associated with M . This also equals the length of any maximal chain of faces, and hence of any maximal chain of idempotents.

(ii) Let S be a closed irreducible subsemigroup of M containing $E(M)$. Then S contains the neutral element of M ; it follows that S is a submonoid of M , and $G(S)$ is a subtorus of T . By (i), we have $\dim(T_0) = \dim(G(S)_0)$. But $G(S)_0 \subset T_0$, and hence equality holds. Taking closures, we obtain that S contains $\overline{T_0}$.

(iii) Let S be a closed irreducible subsemigroup of M having a zero, e . Then e is idempotent; thus, $ee_0 = e_0$. For any $x \in S$, we have $xe = e$ and hence $xe_0 = e_0$. Also, S is a toric monoid by Lemma 3.3. Thus, the closure $\overline{ST_0}$ is a toric monoid, stable by T_0 and contained in M_{e_0} (the fiber of φ at e). Since T has finitely orbits in M , it follows that T_{e_0} has finitely many orbits in M_{e_0} ; since T_0 is a subgroup of finite index in T_{e_0} , we see that T_0 has finitely many orbits in M_{e_0} as well. As a consequence, $\overline{ST_0}$ is the closure of a T_0 -orbit, and thus equals $\overline{e_S T_0}$ for some idempotent e_S of M . But $e_S \in \overline{T_0}$, and hence $\overline{ST_0} \subset \overline{T_0}$. In particular, $S \subset \overline{T_0}$.

(iv) It suffices to show that T_0 is defined over F . For this, we use the bijective correspondence between F -subgroup schemes of T and Γ -stable subgroups of Λ , that associates with any such subgroup scheme T' , the character group of the quotient torus T/H ; then T' is a torus if and only if the corresponding subgroup Λ' is saturated in Λ , i.e., the quotient group Λ/Λ' is torsion-free (these results follow e.g. from [SGA3, Exp. VIII, §2]). Under this correspondence, the isotropy subgroup scheme T_{e_0} is sent to the largest subgroup Λ_{e_0} of Λ contained in the monoid \mathcal{M} (since $\mathcal{O}(T/T_{e_0}) = \mathcal{O}(e_0T) = \mathcal{O}(e_0M)$ is the subalgebra of $\mathcal{O}(M)$ generated by the invertible elements of that algebra). Thus, T_0 corresponds to the smallest saturated subgroup Λ_0 of Λ that contains Λ_{e_0} . Clearly, the action of Γ on Λ stabilizes Λ_{e_0} , and hence Λ_0 . \square

Remark 3.5. The above proof can be shortened by using general structure results for unit dense algebraic monoids (see [Br12, §3.2]). We have chosen to present more self-contained arguments.

3.3 The toric envelope

In this subsection, we return to an irreducible commutative algebraic semigroup S ; we denote by e_0 (resp. e_1) the smallest (resp. largest) idempotent of S , by T the largest subtorus of $G(e_1S)$, and by T_0 the reduced neutral component of the isotropy subgroup scheme T_{e_0} . In view of Proposition 3.2, \overline{T} contains $E(S)$ and is the largest toric submonoid of S ; we denote that submonoid by $TM(S)$ to emphasize its intrinsic nature. We now obtain an intrinsic interpretation of $\overline{T_0}$, thereby completing the proof of Theorem 1.2:

Proposition 3.6. *With the above notation, $\overline{T_0}$ is the smallest closed irreducible subsemigroup of S containing $E(S)$, and also the largest toric subsemigroup of S having a zero. Moreover, $\overline{T_0}$ is defined over F if so is S .*

Proof. Let S' be a closed irreducible subsemigroup of S containing $E(S)$. Then we have $E(S) \subset TM(S') \subset TM(S)$ by Proposition 3.2. Thus, $TM(S')$ contains $\overline{T_0}$ in view of Proposition 3.4. Hence $S' \supset \overline{T_0}$.

Next, let M be a toric subsemigroup of S having a zero. Then $M \subset TM(S)$ by Proposition 3.2, and hence $M \subset \overline{T_0}$ by Proposition 3.4 again. The final assertion follows similarly from that proposition. \square

We denote $\overline{T_0}$ by $TE(S)$, and call it the *toric envelope* of $E(S)$; we may view $TE(S)$ as an algebro-geometric analogue of the finite poset $E(S)$.

Corollary 3.7. *Let S be an algebraic semigroup, x a point of S , and $\langle x \rangle$ the smallest closed subsemigroup of S containing x .*

- (i) $\langle x \rangle$ contains a largest closed toric subsemigroup, $TM(x)$.
- (ii) $E(\langle x \rangle)$ is contained in a smallest closed irreducible subsemigroup of $\langle x \rangle$. Moreover, this subsemigroup, $TE(x)$, is a toric monoid.
- (iii) $TM(x) = TM(x^n)$ and $TE(x) = TE(x^n)$ for any positive integer n .
- (iv) If the algebraic semigroup S and the point x are defined over F , then so are $TM(x)$ and $TE(x)$.

Proof. (i) By [BrRe12, Lem. 2.2], there exists a positive integer n such that $\langle x^n \rangle$ is irreducible. Since $\langle x^n \rangle$ is also commutative, it contains a largest closed toric subsemigroup by Proposition 3.2. But every toric subsemigroup S of $\langle x \rangle$ is contained in $\langle x^n \rangle$. Indeed, we have $S^n \subset \langle x^n \rangle$; moreover, $S = S^n$, since the n th power map of S restricts to a finite surjective homomorphism on any subtorus.

(ii) We claim that $E(\langle x \rangle) = E(\langle x^n \rangle)$ for any positive integer n . Indeed, $E(\langle x^n \rangle)$ is obviously contained in $E(\langle x \rangle)$. For the opposite inclusion, note that for any $y \in \langle x \rangle$, we have $y^n \in \langle x^n \rangle$. Taking y idempotent yields the claim.

Now choose n as in (i); then the desired statement follows from Proposition 3.6 in view of the claim.

(iii) is proved similarly; it implies (iv) in view of Propositions 3.2 and 3.6 again. \square

Remark 3.8. When the algebraic semigroup S is irreducible and commutative, we clearly have $TM(x) \subset TM(S)$ and $TE(x) \subset TE(S)$ for all $x \in S$. Moreover, if the field k is not locally finite (that is, k is not the algebraic closure of a finite field), then there exists $x \in S$ such that $TM(x) = TM(S)$ and $TE(x) = TE(S)$: indeed, the torus $T = G(TM(S))$ has a point x which generates a dense subgroup, and then $\langle x \rangle = TM(S)$.

On the other hand, if k is locally finite, then any algebraic semigroup S is defined over some finite field \mathbb{F}_q . Hence S is the union of the finite subsemigroups $S(\mathbb{F}_{q^n})$, where $n \geq 1$; it follows that $TM(x) = TE(x)$ consists of a unique point, for any $x \in S$.

Example 3.9. Let S be a linear algebraic semigroup, i.e., S is isomorphic to a closed subsemigroup of $\text{End}(V)$ for some finite-dimensional vector space V . Given $x \in S$, we describe the combinatorial data of the toric monoid $TM(x)$ in terms of the spectrum of x (viewed as a linear operator on V).

Consider the decomposition of V into generalized eigenspaces for x ,

$$V = \bigoplus_{\lambda} V_{\lambda},$$

where λ runs over the spectrum. Since x acts nilpotently on V_0 and $TM(x) = TM(x^n)$ for any positive integer n , we may assume that x acts trivially on V_0 . Then we may replace V with $\bigoplus_{\lambda \neq 0} V_{\lambda}$, and hence assume that $x \in \text{GL}(V)$. In that case, $\langle x \rangle \cap \text{GL}(V)$ is a closed subsemigroup of the algebraic group $\text{GL}(V)$, and hence a closed subgroup; in particular, $\text{id}_V \in \langle x \rangle$. So $\langle x \rangle$ is a closed submonoid of $\text{End}(V)$, and $G(\langle x \rangle) = \langle x \rangle \cap \text{GL}(V)$. Thus, $TM(x)$ is the closure of the largest subtorus, T , of the commutative linear algebraic group $G := G(\langle x \rangle)$.

In view of the Jordan decompositions $x = x_s x_u$ and $G = G_s \times G_u$, we see that T is the largest subtorus of $G(\langle x_s \rangle)$. Thus, we may replace x with x_s and assume that

$$x = \text{diag}(\lambda_1, \dots, \lambda_r),$$

where $\lambda_i \in k^*$ for $i = 1, \dots, r$; we may further assume that $\lambda_1, \dots, \lambda_r$ are pairwise distinct. Then $G(\langle x \rangle)$ is contained in the diagonal torus \mathbb{G}_m^r , and the character group of the quotient torus $\mathbb{G}_m^r / G(\langle x \rangle)$ is the subgroup of $\text{Hom}(\mathbb{G}_m^r, \mathbb{G}_m) \cong \mathbb{Z}^r$ consisting of those tuples (a_1, \dots, a_r) such that $\prod_{i=1}^r \lambda_i^{a_i} = 1$. Via the correspondence between closed subgroups of \mathbb{G}_m^r and subgroups of \mathbb{Z}^r , it follows that the character group of $G(\langle x \rangle)$ is the subgroup of k^* generated by $\lambda_1, \dots, \lambda_r$. As a consequence, *the free abelian group Λ associated with the toric monoid $TM(x)$ is isomorphic to the subgroup of k^* generated by the n th powers of the nonzero eigenvalues, for n sufficiently divisible* (so that this subgroup is indeed free).

Moreover, since the coordinate functions generate the algebra $\mathcal{O}(\langle x \rangle)$ and are eigenvectors of $G(\langle x \rangle)$, we see that *the monoid \mathcal{M} associated with $TM(x)$ is isomorphic to the submonoid of (k, \times) generated by the n th powers of the eigenvalues, for n sufficiently divisible* (so that this monoid embeds indeed into a free abelian group).

Next, we describe the idempotents of $\langle x \rangle$, where x is a diagonal matrix as above. These idempotents are among those of the subalgebra of $\text{End}(V)$ consisting of all diagonal matrices; hence they are of the form

$$e_I := \sum_{i \in I} e_i,$$

where $I \subset \{1, \dots, r\}$, and e_i denotes the projection to the i th coordinate subspace. To determine when $e_I \in \langle x \rangle$, we view \mathcal{M} as the monoid generated by t_1, \dots, t_r , with relations of the form

$$\prod_{a \in A} t_i^{a_i} = \prod_{b \in B} t_j^{b_j},$$

where A, B are disjoint subsets of $\{1, \dots, r\}$ (one of them being possibly empty), and a_i, b_j are positive integers; such relations will be called *primitive*. Since

$$E(\langle x \rangle) = E(TM(x)) = \text{Hom}(\mathcal{M}, \{1, 0\}),$$

it follows that $e_I \in \langle x \rangle$ if and only if either I contains $A \cup B$, or I meets the complements of A and of B .

In particular, the largest idempotent of $\langle x \rangle$ is $e_{1, \dots, r} = \text{id}_V$ (this may of course be seen directly). The smallest idempotent is e_I , where $i \in I$ if and only if the i th coordinate is invertible on $\langle x \rangle$; this is equivalent to the existence of a primitive relation of the form $\prod_{a \in A} t_i^{a_i} = 1$, where $i \in A$.

3.4 Algebraic semigroups with finitely many idempotents

Consider an algebraic semigroup S such that $E(S)$ is finite. Then S has a smallest idempotent, as shown by the proof of Proposition 2.3 (ii). Also, when S is irreducible, it has a largest idempotent by the proof of Proposition 3.1. We now obtain criteria for an idempotent of an algebraic semigroup to be the smallest or the largest one (if they exist):

Proposition 3.10. *Let S be an algebraic semigroup, and $e \in S$ an idempotent.*

- (i) *e is the smallest idempotent if and only if e is central and eS is a group.*
- (ii) *When S is irreducible, e is the largest idempotent if and only if e is central and there exists a positive integer n such that $x^n \in eS$ for all $x \in S$.*

Proof. (i) Assume that e is the smallest idempotent. Then both eS_e and ${}_eSe$ consist of the unique point e . Since e is a minimal idempotent, it follows that $SeS = eSe$ by [Br12, Prop. 2.3.3]. In particular, $xe \in eSe$ for any $x \in S$. As a consequence, $xe = exe$; likewise, $ex = exe$ and hence e is central. Thus, $eS = eSe$; the latter is a group by [loc. cit., Prop. 2.3.2].

To show the converse implication, let $f \in S$ be an idempotent. Then ef is an idempotent of the group eS , and hence $ef = e = fe$.

(ii) If e is the largest idempotent, then eS_e and ${}_eSe$ still consist of the unique point e . By Lemma 2.8, it follows that eSe contains Ue ; likewise, eSe contains eU . Since S is irreducible, eSe contains both Se and eS . Arguing as in (i), this yields that e is central. Also, there exists a positive integer n such that every $x \in S$ satisfies $x^n \in e(x)S$ for some idempotent $e(x)$, by the main result of [BrRe12]. Since $e(x) \leq e$, it follows that $x^n \in eS$.

For the converse implication, let again $f \in S$ be an idempotent. Then $f = f^n \in eS$, and hence $f = fe = ef$; in other words, $f \leq e$. \square

We now show that the structure of an irreducible algebraic monoid having finitely many idempotents reduces somehow to that of a closed irreducible submonoid having a zero and the same idempotents:

Proposition 3.11. *The following conditions are equivalent for an irreducible algebraic semigroup S :*

- (i) $E(S)$ is finite.
- (ii) S has a smallest idempotent, e_0 , and a largest one, e_1 . Moreover, $E(M_S)$ is finite, where M_S denotes the reduced neutral component of the submonoid scheme $e_1 S_{e_0} \subset S$ (with unit e_1 and zero e_0).

Under either condition, $E(S)$ is reduced, central in S , and equals $E(M_S)$.

Proof. (i) \Rightarrow (ii) follows from the discussion preceding Proposition 3.10. For the converse, note that $E(M_S) = E(e_1 S_{e_0})$ as sets, by the definitions of e_0 and e_1 ; also, $E(e_1 S_{e_0}) = E(M_S)$ as sets, in view of Theorem 2.14.

If $E(S)$ is finite, then it is reduced and central in S by Proposition 2.10. Since $E(S) = E(M_S)$ as sets, it follows that this also holds as schemes. \square

The above reduction motivates the following statement, due to Putcha (see [Pu82, Cor. 10]); we present an alternative proof, based on Renner’s construction of the “largest reductive quotient” of an irreducible linear algebraic monoid.

Proposition 3.12. *Let M be an irreducible algebraic monoid having a zero. If $E(M)$ is finite, then $G(M)$ is solvable.*

Proof. By [Re85, Thm. 2.5], there exist an irreducible algebraic monoid M' equipped with a homomorphism of algebraic monoids $\rho : M \rightarrow M'$ that satisfies the following conditions:

- (i) ρ restricts to a surjective homomorphism $G := G(M) \rightarrow G(M') =: G'$ with kernel the unipotent radical, $R_u(G)$.
- (ii) ρ restricts to an isomorphism $\overline{T} \rightarrow \overline{T}'$, where T denotes a maximal subtorus of G , and T' its image under ρ .

As a consequence, G' is reductive, that is, M' is a reductive monoid. Also, since each conjugacy class of idempotents in M (resp. M') meets \overline{T} (resp. \overline{T}') and since every idempotent of M is central, we see that $E(M')$ equals $E(\overline{T}')$ and is contained in the center of M' .

Let \mathcal{C} be the cone associated with the toric monoid \overline{T}' . Then \mathcal{C} is stable under the action of the Weyl group $W' := W(G', T')$ on the character group of T' . Recall from Subsection 3.2 that $E(\overline{T}')$ is in a bijective correspondence with the set of faces of \mathcal{C} ; this correspondence is compatible with the natural actions of W' . But W' acts trivially on the idempotents of \overline{T}' , since they are all central. Thus, W' stabilizes every face of \mathcal{C} . It follows that W' fixes pointwise every extremal ray; hence W' fixes pointwise the whole cone \mathcal{C} . Since the interior of \mathcal{C} is nonempty, W' must be trivial, i.e., G' is a torus. Hence G is solvable. \square

Acknowledgements. Most of this article was written during my participation in February 2013 to the Thematic Program on Torsors, Nonassociative Algebras and Cohomological Invariants, held at the Fields Institute. I warmly thank the organizers of the program for their invitation, and the Fields Institute for providing excellent working conditions. I also thank Mohan Putcha and Wenxue Huang for very helpful e-mail exchanges.

References

- [Br08] M. Brion, *Local structure of algebraic monoids*, Mosc. Math. J. **8**, No. 4 (2008), 647–666.
- [Br12] M. Brion, *On algebraic semigroups and monoids*, preprint, arXiv:1208.0675v3.
- [BrRi07] M. Brion and A. Rittatore, *The structure of normal algebraic monoids*, Semigroup Forum **74**, No. 3 (2007), 410–422.
- [BrRe12] M. Brion and L. Renner, *Algebraic semigroups are strongly π -regular*, preprint, arXiv:1209.2042.
- [Ha77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York, 1977.
- [Hu96a] W. Huang, *Nilpotent algebraic monoids*, J. Algebra **179**, No. 3 (1996), 720–731.
- [Hu96b] W. Huang, *Linear associative algebras with finitely many idempotents*, Linear and Multilinear Algebra **40**, No. 4 (1996), 303–309.
- [Ne92] K.-H. Neeb, *Toric varieties and algebraic monoids*, Sem. Sophus Lie **2**, No. 2 (1992), 159–187.
- [Pu81] M. S. Putcha, *On linear algebraic semigroups. III*, Internat. J. Math. Math. Sci. **4**, No. 4 (1981), 667–690; *corrigendum*, *ibid.*, **5**, No. 1 (1982), 205–207.
- [Pu82] M. S. Putcha, *The group of units of a connected algebraic monoid*, Linear and Multilinear Algebra **12**, No. 1 (1982), 37–50.
- [Pu84] M. S. Putcha, *Algebraic monoids with a dense group of units*, Semigroup Forum **28**, No 1–3 (1984), 365–370.
- [Pu88] M. S. Putcha, *Linear algebraic monoids*, London Mathematical Society Lecture Note Series **133**, Cambridge University Press, 1988.
- [Ro56] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.
- [Re85] L. E. Renner, *Reductive monoids are von Neumann regular*, J. Algebra **93**, No. 2 (1985), 237–245.
- [Re05] L. Renner, *Linear algebraic monoids*, Encyclopedia of Mathematical Sciences **134**, Invariant Theory **5**, Springer-Verlag, 2005.
- [Ri98] A. Rittatore, *Algebraic monoids and group embeddings*, Transform. Groups **3**, No. 4 (1998), 375–396.
- [Ri07] A. Rittatore, *Algebraic monoids with affine unit group are affine*, Transform. Groups **12**, No. 3 (2007), 601–605.

- [SGA3] *Séminaire de Géométrie Algébrique du Bois-Marie 1962/64 (SGA3), Schémas en Groupes II*, Lecture Notes Math. **152**, Springer-Verlag, New York, 1970.
- [Sp98] T. A. Springer, *Linear algebraic groups. Second edition*, Progress in Mathematics **9**, Birkhäuser, Boston, MA, 1998.